Math 146C Homework #2
Solutions To Graded Problems

10.7: **6**,9,**13**,14,**21**
10.5: **3**,6,8,**9**,13,14,20,**22**
10.6: 4,**5**,10,12,**21**
10.8: **1**,3

Section 10.7

Problem 6 - Consider an elastic string of length L whose ends are held fixed. The string is set in motion from its equilibrium position with an initial velocity $u_t(x,0) = g(x)$. In parts (b) and (c) let $L = 10$ and $a = 1$.

(a) Find the displacement $u(x,t)$ given that $g(x) = \begin{cases} \frac{4x}{L} & 0 \le x \le \frac{L}{4} \\ 1 & \frac{L}{4} \le x \le \frac{3L}{4} \\ \frac{4(x-L)}{L} & \frac{3L}{4} \le x \le L \end{cases}$.

(b) Plot $u(x,t)$ versus x for $0 \le x \le 10$ and for several values of t between $t = 0$ and $t = 20$.

(c) Plot $u(x,t)$ versus t for $0 \le t \le 20$ and for several values of x.

Solution -

(a) The motion of the string is governed by the following equations:

1. $a^2 u_{xx} = u_{tt}$
2. $u(0,t) = u(L,t) = 0, \qquad t \ge 0$
3. $u(x,0) = 0, \qquad u_t(x,0) = g(x), \qquad 0 \le x \le L$, where $g(x)$ is the initial velocity of the string at the point x.

and thus the solution is given by $u(x,t) = \sum_{n=1}^{\infty} k_n u_n(x,t) = \sum_{n=1}^{\infty} k_n \sin\frac{n\pi x}{L} \sin\frac{n\pi a t}{L}$.

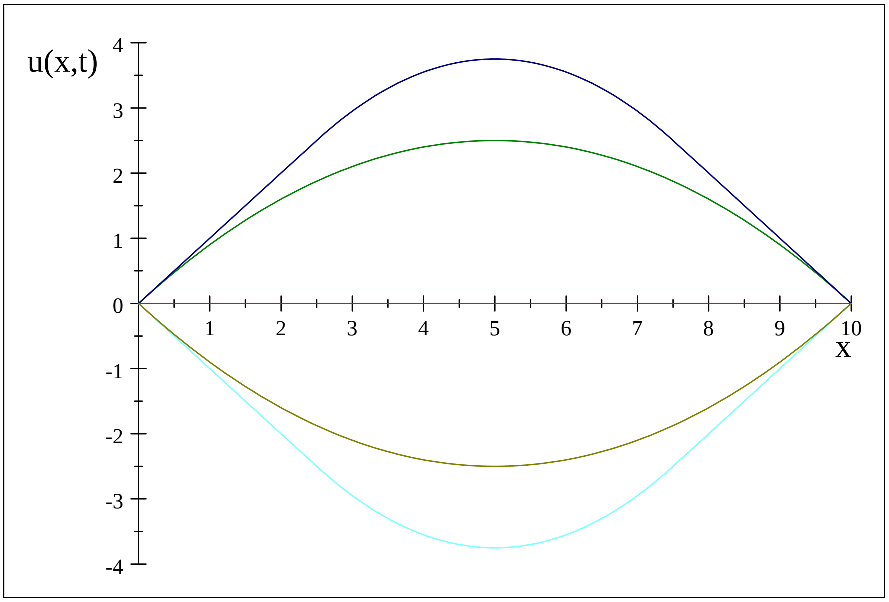
Then $k_n = \frac{2}{n\pi a}\left(\int_0^{\frac{L}{4}} \frac{4x}{L}\sin\frac{n\pi x}{L}dx + \int_{\frac{L}{4}}^{\frac{3L}{4}} \sin\frac{n\pi x}{L}dx + \int_{\frac{3L}{4}}^{L} \frac{4(x-L)}{L}\sin\frac{n\pi x}{L}dx\right) \overset{Integration\ By\ Parts}{=}$ $\frac{8L}{n^3\pi^3 a}\left(\sin\frac{n\pi}{4} + \sin\frac{3n\pi}{4}\right)$.

So then $u(x,t)$ is given by:

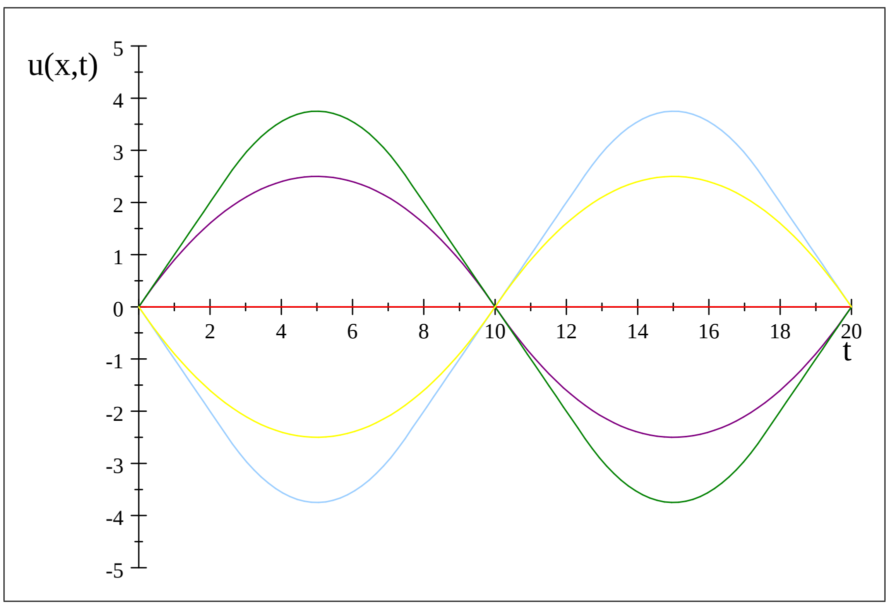
$$u(x,t) = \sum_{n=1}^{\infty} \frac{8L}{n^3\pi^3 a}\left(\sin\frac{n\pi}{4} + \sin\frac{3n\pi}{4}\right)\sin\frac{n\pi x}{L}\sin\frac{n\pi a t}{L} = \frac{8L}{\pi^3 a}\sum_{n=1}^{\infty} \frac{\left(\sin\frac{n\pi}{4} + \sin\frac{3n\pi}{4}\right)}{n^3}\sin\frac{n\pi x}{L}\sin\frac{n\pi a t}{L}$$

`Note:`In parts (b) and (c) $\qquad u(x,t) = \frac{80}{\pi^3}\sum_{n=1}^{\infty} \frac{\left(\sin\frac{n\pi}{4} + \sin\frac{3n\pi}{4}\right)}{n^3}\sin\frac{n\pi x}{10}\sin\frac{n\pi t}{10}$

(b) $u(x,0), u(x,10), u(x,20)$ in light red, $u(x,2.5), u(x,7.5)$ in green, $u(x,5)$ in purple, $u(x,12.5), u(x,17.5)$ in brown, $u(x,15)$ in cyan.



(c) $u(0,t), u(10,t), u(20,t)$ in light red, $u(2.5,t), u(7.5,t)$ in purple, $u(5,t)$ in green, $u(12.5,t), u(17.5,t)$ in yellow, $u(15,t)$ in light blue.



Problem 13 - Show that the wave equation

$$a^2 u_{xx} = u_{tt}$$

can be reduced to the form $u_{\xi\eta} = 0$ by the change of variables $\xi = x - at$, $\eta = x + at$. Show that $u(x,t)$ can be written as

$$u(x,t) = \phi(x - at) + \psi(x + at),$$

where ϕ and ψ are arbitrary functions.

Solution - Recall the chain rule for partial derivatives: $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$
First notice that $\xi_x = \eta_x = 1$, and $\xi_t = -a = -\eta_t$
Using the chain rule to differentiate with respect to the x variable we get:

$$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$$

Taking a second derivative we get:

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

Now take the first and second derivatives with respect to t:

$$\begin{gathered} u_t = u_\xi \xi_t + u_\eta \eta_t = -au_\xi + a\eta_t \\ u_{tt} = a^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}). \end{gathered}$$

So plugging u_{xx} and u_{tt} into the wave equation yields:

$$\begin{gathered} a^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) = a^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) \\ \iff a^2u_{\xi\xi} + 2a^2u_{\xi\eta} + a^2u_{\eta\eta} = a^2u_{\xi\xi} - 2a^2u_{\xi\eta} + a^2u_{\eta\eta} \\ \iff 4a^2u_{\xi\eta} = 0 \\ \iff u_{\xi\eta} = 0. \end{gathered}$$

Now integrate both sides of $u_{\xi\eta} = 0$ with respect to η to get:

$$u_\xi(\xi,\eta) = \rho(\xi),\ \rho \text{ is an arbitrary function of } \xi.$$

Then integrating both sides of $u_\xi(\xi,\eta) = \rho(\xi)$ with respect to ξ we get:

$$u(\xi,\eta) = \int \rho(\xi)\, d\xi + \psi(\eta) = \phi(\xi) + \psi(\eta).$$

Thus:

$$u(x,t) = u(\xi,\eta) = \phi(x - at) + \psi(c + at).$$

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Problem 21 - The motion of a circular elastic membrane, such as a drumhead, is governed by the two dimensional wave equation in polar coordinates

$$u_{rr} + \tfrac{1}{r}u_r + \tfrac{1}{r^2}u_{\theta\theta} = a^{-2}u_{tt}$$

Assuming that $u(r,\theta,t) = R(r)\,\Theta(\theta)\,T(t)$, find ordinary differential equations satisfied by $R(r)$, $\Theta(\theta)$, and $T(t)$.

Solution - First we need all of the partial derivatives:

$$\begin{aligned} u_r &= R'(r)\,\Theta(\theta)\,T(t) \\ u_{rr} &= R''(r)\,\Theta(\theta)\,T(t) \\ u_{\theta\theta} &= R(r)\,\Theta''(\theta)\,T(t) \\ u_{tt} &= R(r)\,\Theta(\theta)\,T''(t). \end{aligned}$$

Now we plug these into the equation to get:

$$R''(r)\,\Theta(\theta)\,T(t) + \tfrac{1}{r}R'(r)\,\Theta(\theta)\,T(t) + \tfrac{1}{r^2}R(r)\,\Theta''(\theta)\,T(t) = a^{-2}R(r)\,\Theta(\theta)\,T''(t)$$

Divide through by $T(t)$:

$$\Longrightarrow R''(r)\,\Theta(\theta) + \tfrac{1}{r}R'(r)\,\Theta(\theta) + \tfrac{1}{r^2}R(r)\,\Theta''(\theta) = a^{-2}R(r)\,\Theta(\theta)\,\tfrac{T''(t)}{T(t)}$$

Now divide by $R(r)\,\Theta(\theta)$ on both sides:

$$\Longrightarrow \tfrac{R''(r)}{R(r)} + \tfrac{1}{r}\tfrac{R'(r)}{R(r)} + \tfrac{1}{r^2}\tfrac{\Theta''(\theta)}{\Theta(\theta)} = a^{-2}\tfrac{T''(t)}{T(t)}$$

In order for this equation to be valid for $0 < r < r_0$, $0 \leq \theta < 2\pi$, $t > 0$, it is necessary that both sides equal the same constant, call it $-\sigma$.

$$\tfrac{R''(r)}{R(r)} + \tfrac{1}{r}\tfrac{R'(r)}{R(r)} + \tfrac{1}{r^2}\tfrac{\Theta''(\theta)}{\Theta(\theta)} = a^{-2}\tfrac{T''(t)}{T(t)} = -\sigma$$

Which yields the system:

$$\begin{aligned} &\left\{ \begin{array}{c} \tfrac{R''(r)}{R(r)} + \tfrac{1}{r}\tfrac{R'(r)}{R(r)} + \tfrac{1}{r^2}\tfrac{\Theta''(\theta)}{\Theta(\theta)} = -\sigma \\ a^{-2}\tfrac{T''(t)}{T(t)} = -\sigma \end{array} \right. \\ \Longrightarrow &\left\{ \begin{array}{c} r^2\tfrac{R''(r)}{R(r)} + r\tfrac{R'(r)}{R(r)} + \tfrac{\Theta''(\theta)}{\Theta(\theta)} = -\sigma r^2 \\ T''(t) = -a^2\sigma T(t) \end{array} \right. \end{aligned}$$

Concentrating on the first equation: $r^2\tfrac{R''(r)}{R(r)} + r\tfrac{R'(r)}{R(r)} + \tfrac{\Theta''(\theta)}{\Theta(\theta)} = -\sigma r^2$

$$r^2\tfrac{R''(r)}{R(r)} + r\tfrac{R'(r)}{R(r)} + \sigma r^2 = -\tfrac{\Theta''(\theta)}{\Theta(\theta)}$$

Demanding that both sides equal the same constant we get:

$$r^2\tfrac{R''(r)}{R(r)} + r\tfrac{R'(r)}{R(r)} + \sigma r^2 = -\tfrac{\Theta''(\theta)}{\Theta(\theta)} = \delta$$

So now we can separate these as well to get the system:

$$\left\{\begin{array}{c} r^2\frac{R''(r)}{R(r)}+r\frac{R'(r)}{R(r)}+\sigma r^2=\delta \\ -\frac{\Theta''(\theta)}{\Theta(\theta)}=\delta \end{array}\right.$$
$$\Longrightarrow \left\{\begin{array}{c} r^2R''\left(r\right)+rR'\left(r\right)+\sigma r^2R\left(r\right)=\delta R\left(r\right) \\ \Theta''\left(\theta\right)=-\delta\Theta\left(\theta\right) \end{array}\right.$$
$$\Longrightarrow \left\{\begin{array}{c} r^2R''\left(r\right)+rR'\left(r\right)+\left(\sigma r^2-\delta\right)R\left(r\right)=0 \\ \Theta''\left(\theta\right)+\delta\Theta\left(\theta\right)=0 \end{array}\right.$$

Since the circular membrane is continuous, we must have $\Theta\left(2\pi\right)=\Theta\left(0\right)$, which requires that $\delta=\mu^2$, where $\mu\in\mathbb{N}_0$. $\Theta\left(2\pi\right)=\Theta\left(0\right)$ is also known as the periodicity condition. Since we want that solutions vary periodically in time, we need that $\sigma>0$, so let $\sigma=\lambda^2$.

So the system of three equations looks like:

$$\boxed{\left\{\begin{array}{c} r^2R''\left(r\right)+rR'\left(r\right)+\left(\lambda^2r^2-\mu^2\right)R\left(r\right)=0 \\ \Theta''\left(\theta\right)+\mu^2\Theta\left(\theta\right)=0 \\ T''\left(t\right)+a^2\lambda^2T\left(t\right)=0 \end{array}\right.}$$

Section 10.5

Problem 3 - Determine whether the following partial differential equation is separable. If so, find the pair of ordinary differential equations it separates into.

$$u_{xx}+u_{xt}+u_t=0$$

Solution - Assume $u=XT$.
Then: $u_x=X'T$, $u_{xx}=X''T$, $u_t=XT'$, $u_{xt}=X'T'$.
Plugging in:

$$X''T+X'T'+XT'=0$$
$$\Longleftrightarrow X''T+\left(X'+X\right)T'=0$$
$$\Longleftrightarrow X''T=-\left(X'+X\right)T'$$
$$\Longleftrightarrow \frac{X''}{\left(X'+X\right)}=\frac{-T'}{T}$$

We demand that both sides of the equation be equal to the same constant:

$$\frac{X''}{\left(X'+X\right)}=\frac{-T'}{T}=\lambda$$
$$\Longrightarrow \left\{\begin{array}{c} \frac{X''}{\left(X'+X\right)}=\lambda \\ \frac{-T'}{T}=\lambda \end{array}\right.$$
$$\Longleftrightarrow \boxed{\left\{\begin{array}{c} X''-\lambda\left(X'+X\right)=0 \\ T'+\lambda T=0 \end{array}\right.}$$

Problem 9 - Consider the conduction of heat in a rod 40 cm in length whose ends are maintained at 0°C for all $t > 0$. Find an expression for $u(x,t)$ if the initial temperature distribution in the rod is $u(x,0) = 50$, $0 < x < 40$. Suppose that $\alpha^2 = 1$.

Solution - For simplicity I like to write the equation in a "system" form:

$$\begin{cases} u_{xx} = u_t & 0 < x < 40,\ t > 0 \\ u(0,t) = u(40,t) = 0 & t > 0 \\ u(x,0) = 50 = f(x) & 0 < x < 40 \end{cases}$$

Assuming that $u = XT$, and following the method of separation of variables we get the eigenfunctions:

$$X_n = \sin\frac{n\pi x}{40}$$

with eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{1600}$$

as well as the temporal equations

$$T_n = e^{-\frac{n^2\pi^2 t}{1600}}$$

which I assume that you are perfectly able to calculate on your own by now. Now the solution is given by:

$$u(x,t) = \sum_{n=1}^{\infty} c_n T_n X_n = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2 t}{1600}} \sin\frac{n\pi x}{40}$$

where

$$c_n = \tfrac{2}{L}\int_0^L f(x)\sin\tfrac{n\pi x}{L}dx = \tfrac{2}{40}\int_0^{40} 50\sin\tfrac{n\pi x}{40}dx = \tfrac{5}{2}\int_0^{40}\sin\tfrac{n\pi x}{40}dx = 100\tfrac{1-\cos n\pi}{n\pi}.$$

And thus the final solution is:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} 100\tfrac{1-\cos n\pi}{n\pi} e^{-\frac{n^2\pi^2 t}{1600}} \sin\tfrac{n\pi x}{40} = \tfrac{100}{\pi}\sum_{n=1}^{\infty}\tfrac{1-\cos n\pi}{n} e^{-\frac{n^2\pi^2 t}{1600}} \sin\tfrac{n\pi x}{40}}$$

Problem 22 - The heat conduction equation in two space dimensions is

$$\alpha^2(u_{xx} + u_{yy}) = u_t.$$

Assuming that $u(x,y,t) = X(x)Y(y)T(t)$, find ordinary differential equations that are satisfied by $X(x)$, $Y(y)$, and $T(t)$.

Solution - Find the necessary partial derivatives:

$$u_{xx} = X''YT$$
$$u_{yy} = XY''T$$
$$u_t = XYT'$$

Now we plug this into the equation:

$$\alpha^2 \left(X''YT + XY''T\right) = XYT'$$
$$\Longrightarrow \alpha^2 X''YT + \alpha^2 XY''T = XYT'$$

And now divide through by XYT:

$$\tfrac{X''}{X} + \tfrac{Y''}{Y} = \tfrac{T'}{\alpha^2 T}$$

And mandate that both sides equal the same constant:

$$\tfrac{X''}{X} + \tfrac{Y''}{Y} = \tfrac{T'}{\alpha^2 T} = \kappa$$

So separation yields:

$$\begin{cases} \tfrac{X''}{X} + \tfrac{Y''}{Y} = \kappa \\ \tfrac{T'}{\alpha^2 T} = \kappa \end{cases}$$
$$\Longrightarrow \begin{cases} \tfrac{X''}{X} + \tfrac{Y''}{Y} = \kappa \\ \tfrac{T'}{\alpha^2 T} = \kappa \end{cases}$$
$$\Longrightarrow \begin{cases} \tfrac{X''}{X} = \kappa - \tfrac{Y''}{Y} \\ T' - \kappa\alpha^2 T = 0 \end{cases}$$

And focusing on the top equation: $\tfrac{X''}{X} = \kappa - \tfrac{Y''}{Y}$

This equation is already separated so impose that both sides equal the same constant to get:

$$\tfrac{X''}{X} = \kappa - \tfrac{Y''}{Y} = \zeta$$

Which yields the system:

$$\begin{cases} \tfrac{X''}{X} = \zeta \\ \kappa - \tfrac{Y''}{Y} = \zeta \end{cases}$$
$$\Longleftrightarrow \begin{cases} X'' = \zeta X \\ \kappa Y - Y'' = \zeta Y \end{cases}$$
$$\Longleftrightarrow \begin{cases} X'' - \zeta X = 0 \\ Y'' - (\kappa - \zeta)\, Y = 0 \end{cases}$$

In order for $T' - \kappa\alpha^2 T = 0$ to have solutions that remain bounded as $t \to \infty$ we need that $\kappa < 0$. Thus we can let $\kappa = -\lambda^2$ and get $T' + \lambda^2\alpha^2 T = 0$. For $X'' - \zeta X = 0$ and the homogeneous boundary conditions, it is necessary that $\zeta < 0$, so let $\zeta = -\mu^2$, then $X'' + \mu^2 X = 0$. And we also have that $Y'' + \left(\lambda^2 - \mu^2\right) Y = 0$ by choice.

So the whole system is:

$$\boxed{\begin{cases} X'' + \mu^2 X = 0 \\ Y'' + \left(\lambda^2 - \mu^2\right) Y = 0 \\ T' + \lambda^2 \alpha^2 T = 0 \end{cases}}$$

Section 10.6

Problem 5 - Find the steady-state solution of the heat conduction equation $\alpha^2 u_{xx} = u_t$ that satisfies the boundary conditions: $u(0,t) = 0$, $u_x(L,t) = 0$.

Solution -The steady-state solution is the solution v that satisfies

$$v''(x) = 0$$
$$v(0) = v'(L) = 0.$$

So since $v(x) = ax + b$ and $v'(x) = a$, we have that :
$v(0) = b = 0 \Longrightarrow b = 0$
and
$v'(0) = a = 0 \Longrightarrow a = 0$
$\therefore$ The steady-state part of the solution is given by $\boxed{v(x) = 0}$

Problem 21 - Consider the heat conduction problem in a bar that is in thermal contact with an external heat source or sink. Then the modified heat conduction equation is

$$u_t = \alpha^2 u_{xx} + s(x) \qquad (1)$$

where the term $s(x)$ describes the effect of external agency; $s(x)$ is positive for a source and negative for a sink. Suppose that the boundary conditions are

$$u(0,t) = T_1, \qquad u(L,t) = T_2 \qquad (2)$$

and the initial condition is

$$u(x,0) = f(x). \qquad (3)$$

Write $u(x,t) = v(x) + w(x,t)$, where v and w are the steady state and transient parts of the solution, respectively. State the boundary value problems that $v(x)$ and $w(x,t)$, respectively, satisfy. Observe that the problem for w is the fundamental heat conduction problem discussed in Section 10.5, with a modified initial temperature distribution.

Solution - `I would like to note that this problem is good (and in my view important) to do because it makes doing many of the inhomogeneous heat equation problems easier to solve, and besides it is a nice conceptual problem!`

First we tackle the steady-state part of the solution.
So for this part we let $u(x,t) = v(x)$.
Then equation (1) in this problem becomes:

$$0 = \alpha^2 v''(x) + s(x)$$

And equations (2) become:

$$u(0,t) = v(0) = T_1, \qquad u(L,t) = v(L) = T_2$$

Now the transient part we use $u(x,t) = v(x) + w(x,t)$.
So now equation (1) becomes:

$$w_t = \alpha^2 (v'' + w_{xx}) + s(x) \qquad (1)'$$

Equations (2) become:

$$\begin{gathered} u(0,t) = v(0) + w(0,t) = T_1 + w(0,t) = T_1 \\ u(L,t) = v(L) + w(L,t) = T_2 + w(L,t) = T_2 \end{gathered}$$

Lastly equation (3) becomes:

$$\begin{gathered} u(x,0) = v(x) + w(x,0) = f(x) \\ \Longleftrightarrow w(x,0) = f(x) - v(x) \end{gathered}$$

Rearranging and simpifying equation $(1)'$:

$$\begin{gathered} w_t = \alpha^2 (v'' + w_{xx}) + s(x) = \left(\alpha^2 v'' + s(x)\right) + \alpha^2 w_{xx} = 0 + \alpha^2 w_{xx} = \alpha^2 w_{xx} \\ \Longleftrightarrow w_t = \alpha^2 w_{xx} \end{gathered}$$

So the steady-state system (for v) is given by:

$$\boxed{\left\{ \begin{array}{c} \alpha^2 v''(x) + s(x) = 0 \\ v(0) = T_1, \qquad v(L) = T_2 \end{array} \right.}$$

And the transient system (for w) is given by:

$$\boxed{\left\{ \begin{array}{c} w_t = \alpha^2 w_{xx} \\ w(0,t) = w(L,t) = 0 \\ w(x,0) = f(x) - v(x) \end{array} \right.}$$

Section 10.8

Problem 1 -
(a) Find the solution $u(x,y)$ of Laplace's equation in the rectangle $0 < x < a, 0 < y < b$, that satisfies the boundary conditions

$$u(0,y)=0, \qquad u(a,y)=0, \qquad 0<y<b,$$
$$u(x,0)=0, \qquad u(x,b)=g(x), \qquad 0\le x\le a.$$

(b) Find the solution if

$$g(x)=\begin{cases} x & 0\le x\le \frac{a}{2} \\ a-x & \frac{a}{2}\le x\le a \end{cases}.$$

(c) For $a=3$ and $b=1$ plot u versus x for several values of y and also plot u versus y for several values of x.

(d) Plot u versus both x and y in three dimensions.

Solution -

(a) Using the method of separation of variables by letting $u(x,y)=XY$ and plugging it into the Laplace equation:

$$u_{xx}+u_{yy}=0$$

we end up with two ODEs:

$$\begin{cases} X''-\zeta X=0 \\ Y''+\zeta Y=0 \end{cases}$$

The boundary conditions $u(0,y)=0$, $u(a,y)=0 \Longrightarrow X(0)=0$, $X(a)=0$ respectively.

The boundary condition $u(x,0)=0 \Longrightarrow Y(0)=0$.

Solving

$$\begin{cases} X''-\zeta X=0 \\ X(0)=0,\ X(a)=0 \end{cases} \qquad (\bigstar)$$

gives us that we only have nontrivial solutions if $\zeta=-\left(\frac{n\pi}{a}\right)^2$, $n\in\mathbb{N}$. So the solutions to $(\bigstar)$ are proportional to $\sin\left(\frac{n\pi x}{a}\right)$.

Now solving

$$\begin{cases} Y''+\zeta Y=0 \\ Y(0)=0 \end{cases} \qquad (✠)$$

tells us that the solutions to (✠) must be proportional to $\sinh\left(\frac{n\pi y}{a}\right)$.

So then the fundamental solutions u_n are given by:

$$u_n(x,y)=X_n(x)Y_n(y)=\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)$$

Assume that $u(x,y)=\sum_{n=1}^{\infty}c_n u_n(x,y)=\sum_{n=1}^{\infty}c_n\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)$, where the c_n come from the Fourier coefficients of the boundary condition $u(x,b)=g(x)$. So then

$$c_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

Thus the solution is:

$$\boxed{\begin{cases} u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right) \\ c_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)dx \end{cases}}$$

(b) Substitute the given $g(x)$ into the equation for c_n and we get:

$$c_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)dx = \frac{2}{a}\left(\int_0^{\frac{a}{2}} x\sin\left(\frac{n\pi x}{a}\right)dx + \int_{\frac{a}{2}}^{a}(a-x)\sin\left(\frac{n\pi x}{a}\right)dx\right)$$

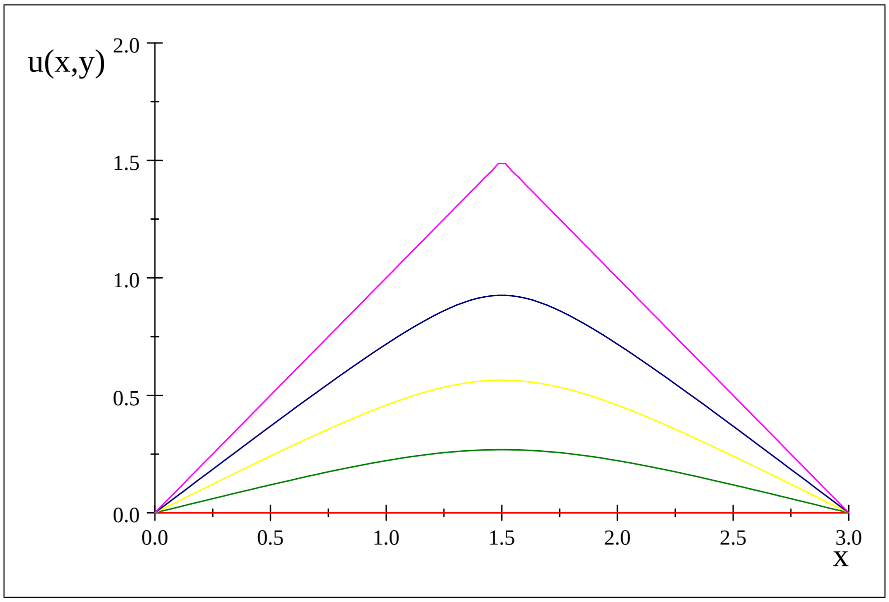
$$\overset{Integration\ By\ Parts}{=} \frac{4a\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}$$

So we have that $c_n = \frac{4a}{\pi^2}\left(\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2\sinh\left(\frac{n\pi b}{a}\right)}\right)$.

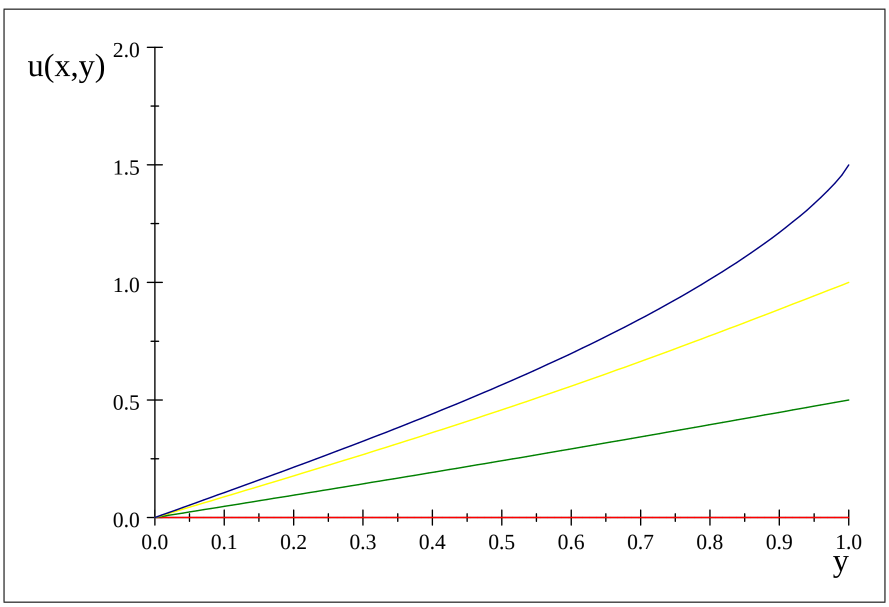
Thus:

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right) = \sum_{n=1}^{\infty}\frac{4a}{\pi^2}\left(\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2\sinh\left(\frac{n\pi b}{a}\right)}\right)\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)}$$

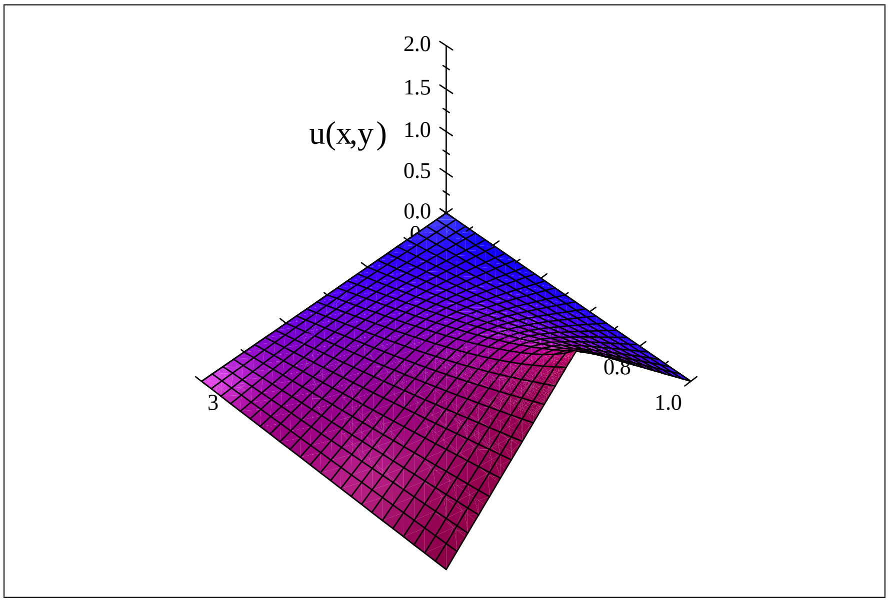
(c) $u(x,0)$ in light red, $u(x,.25)$ in green, $u(x,.5)$ in yellow, $u(x,.75)$ in blue, $u(x,1)$ in magenta



$u(0,y)$, $u(3,y)$ in light red, $u(.5,y)$, $u(2.5,y)$ in green, $u(1,y)$, $u(2,y)$ in yellow, $u(1.5,y)$ in blue



(d) Plot u versus both x and y in three dimensions.


2.0
1.5
u(x,y)
1.0
0.5
0.0
0.8
1.0
3